A Case Study of Different Kernel Applications in Quasi-Geoid Determination

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Abstract. Stokes´s integral is always carried out up to a certain distance from the computation point, a limited cap size. The consequence of this procedure is the origin of the truncation error, which was first studied by Molodenskÿ. The recent use of geopotential models that take into consideration the remote zone contribution in the quasi-geoid estimation drove a different philosophy for modifying the kernel. Software has been developed at EPUSP for the numerical integration of Stokes's integral using the Vanícĕk/Kleusberg approach for the modified kernel. Recently, Featherstone introduced different kernel approaches to the 1D-FFT software developed at the University of Calgary. Different comparisons were carried out in the state of São Paulo, Brazil, using the different options of 1D-FFT software provided by Featherstone and the EPUSP software. The results are presented and analysed.

Keywords:, Geoid, Geopotential, Truncation Error, Modified Kernels

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1 Introduction

In the late 1950's Molodenskÿ developed the idea to minimize the truncation error on the geoid height due to the limitation of Stokes's integral to a cap size ψ_o. His approach yielded to the so- called Molodenskÿ coefficients Q_n computed as a function of the Legendre polynomials as: [Molodenskÿ et al., 1962]

$$Q_n(\psi_O) = \int_{\psi_o}^{\pi} S(\psi)\, P_n(\cos\psi) \sin\psi d\psi \qquad (1)$$

where $S(\psi)$ is the Stokes´s function and ψ the spherical distance.

More recently, with the possibility of using global geopotential models, new attention was paid to the modification of the Stoke's kernel. In fact, the original reason of the truncation error was due to the negligence of the long-wavelength influence of the distant zone (beyond ψ_o) on the geoid estimation. This can easily be dealt with using geopotential models. The lower order coefficients of the model provide the long- wavelength component of the geoid, which is much of the influence of the distant zone. Due to this fact, a new reason exists to modify the Stokes's kernel explain.

2 Different kernel modification

The geoid height can be split into two components, e.g., [Blitzkow, 1986]:

$$N(\theta,\lambda) = -R\sum_{n=2}^{s}\sum_{m=0}^{n}(\overline{J'}_{nm}\overline{Y}^{c}_{nm} + \overline{K'}_{nm}\overline{Y}^{s}_{nm}) - $$
$$- R\sum_{n=s+1}^{\infty}\sum_{m=0}^{n}(\overline{J'}_{nm}\overline{Y}^{c}_{nm} + \overline{K'}_{nm}\overline{Y}^{s}_{nm}) \qquad (2)$$

or in a short form:

$$N(\theta,\lambda) = N_S(\theta,\lambda) + \delta N_S(\theta,\lambda) \qquad (3)$$

The first term $N_s(\theta,\lambda)$ can easily be computed with a geopotential model, e.g., EGM96. The short-wavelength compo-nent $\delta N(\theta, \lambda)$ is estimated from some local gravity anomalies reduced by the geopotential model and using a modified Stokes´s kernel through the following equation:

$$\delta N_s(\theta,\lambda)=\frac{R}{4\pi\gamma}\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\psi_o} S^M(\psi)\Delta g^s \sin\psi d\psi d\alpha \qquad (4)$$

where $S^M(\psi)$ is the modified kernel.

Different ideas have been developed since the simple approach presented by Wong and Gore (1969), which involves removing the low-degree terms from the kernel, according to:

$$S^{WG}(\psi) = S(\psi) - \sum_{k=2}^{s} \frac{2k+1}{k-1} P_k(\psi)$$

for $(0 < \psi \leq \psi_o)$ (5)

Equation (5) is also known as spheroidal Stokes´s kernel. (Vanicěk and Sjöberg, 1991).

Another idea came from Meissl, (1971) who simply subtracts the value of Stokes's kernel at the truncation distance ψ_o from the kernel itself. Thus, Meissl kernel reads:

$$S^{ME}(\psi) = S(\psi) - S(\psi_o)$$

for $(0 < \psi \leq \psi_o)$ (6)

An approach that addressed a better modification of the kernel is the so called Molodenskÿ-modified spherical kernel which is described in Vaníček and Kleusberg (1987), Blitzkow (1996) and revised by [Vaníček and Featherstone, 1998). The formula is:

$$S^{VK}(\psi) = S^{WG}(\psi) - \sum_{k=2}^{s} \frac{2k+1}{2} t_k(\psi_o) P_k(\psi) \qquad (7)$$

which is dependent on the $t_k(\psi_o)$ coefficients to be determined and $S^{WG}(\psi)$ is given by Eq. (5). The following system of linear equations provides the necessary tool for the application of the least squares principle for the estimation of the coefficients $t_k(\psi_o)$:

$$\sum_{k=2}^{s} \frac{2k+1}{2} t_k(\psi_o) e_{nk}(\psi_o) = Q_n^s(\psi_o)$$

$$= Q_n(\psi_o) - \sum_{k=2}^{s} \frac{2k+1}{k-1} e_{nk}(\psi_o) \qquad (8)$$

where:

$$e_{nk}(\psi_o) = \int_{\psi_o}^{\pi} P_n(\cos\psi) P_k(\cos\psi) \sin\psi d\psi \qquad (9)$$

which can be computed through the integral by some algorithm, (e.g., Paul, 1973). The last term of Eq.(8) can also be presented as the integral:

$$Q_n^s(\psi_o) = \int_{\psi_o}^{\pi} S^{WG}(\psi) P_n(\cos\psi) \sin(\psi) d\psi \qquad (10)$$

Equation (8) has the form :

$$\overline{L} = A\overline{X} \qquad (11)$$

$\overline{L}$ and $\overline{X}$ being the following vectors:

$$\overline{L} = [Q_n^s(\psi_o)]^T \text{ and } \overline{X} = [t_k(\psi_o)]^T \qquad (12)$$

Featherstone et al., (1998) proposed a hybrid modification combining the Vaníček and Kleusberg kernel with Meissl's idea. The basic equation is:

$$S^{FEO}(\psi) = S^{VK}(\psi) - S^{VK}(\psi_o) \qquad (13)$$

The limit of integration is loosely dependent of the maximum degree and order that is selected for the geopotential model. As an example, if s = 60, ψ_o will be 3^o and a total of 59 t_k coefficients have to be estimated if the Stokes´s integral is applied to a properly scaled gravity model in a geocentric coordinate system. Software called STKMOD.FOR is available, if requested, from the authors for the computation of the coefficients. It was originally developed at the, University of New Brunswick, and modified at EPUSP in 1996 and in 2002.

3 Geoid computation softwares

The alternative of kernel modification represented by Eq. (7) is implemented since 1990 in the sofware STKMOD.FOR as mentioned. On the other hand, Featherstone and Sideris, (1997), described the modifications introduced in the 1D-FFT software developed at the University of Calgary, in order to implement Eqs. (5), (6) and (13). A copy of the new version of the software was provided by W. E. Featherstone, Curtin University of Technology, and set up at a Sun workstation at EPUSP.

A set of mean Helmert gravity anomalies in an area limited by 9º S and 36º S of latitude and 64º W and 34º W of longitude were used for the computations of this paper, and the results are limited to an area internal to that block, with 5º difference for each limit to avoid edge effects. Four different computations were carried out, three with 1D-FFT software and a fourth one with STOKESMOD.FOR, developed at EPUSP. The results are described in the next paragraphs.

The remove-restore technique was used for the computations taking EGM96 up to degree and order 50 as the reference field. In this way, the limit of integration selected was $\psi_o = 4^o$. The reference field has been removed from 10´ mean Helmert anomalies derived from point anomalies. The 10´ mean anomalies are on a grid with 162 rows and 180 columns. A set of 89 GPS leveled points were also used for comparisons.

The RMS differences have been computed according to:

$$\Delta N = N_S - N_E \qquad (14)$$

where N_S and N_E are geoid heights derived from GPS and Stokes´s integral respectively.

and

$$\sigma_{\Delta N} = \sqrt{\frac{\Sigma\,(\Delta N\;)^2}{n\,(n-1)}} \qquad (15)$$

4 Results

The following experiments were carried out using the numerical integration of Stokes integral and 1D-FFT. The first one will be mentioned from now on as experiment 1 (DB-VK); 1D-FFT has three possibilities: Vaniček/Kleusberg as experiment 2 (FFT-VK), Wong-Gore as experiment 3 (FFT-WG) and Featherstone et al (1998) as experiment 4 (FFT-FEO). Tables 1 and 2 show the results in terms of RMS difference, mean difference and maximum and minimum values. Figures 1 to 6 present the histograms accordingly.

As it is shown in Figure 1, the approaches of experiments 2 and 4 are similar. This is also shown by the second row of Table 1. Experiment 3 differs from both as it can be seen from the RMS difference, rows 3 and 4 of Table 1, and Figures 2 and 3. In any case, the difference is smaller than 1 m. Experiment 1 has a desagreement with all the others, but the desagreement is worse with experiment 3 (see Table 2, second column). In any case the differences are on the order of sub-meter.

Table 1 Differences of the comparisons related to experiments with FFT.

Experiments	RMS (m)	Mean (m)	Maximun (m)	Mínimum (m)
FFT-FEO and FFT-VK	0.01	0.01	0.02	-0.01
FFT-WG and FFT-VK	0.32	0.24	0.86	-0.19
FFT-FEO and FFT-WG	0.32	0.24	0.84	-019

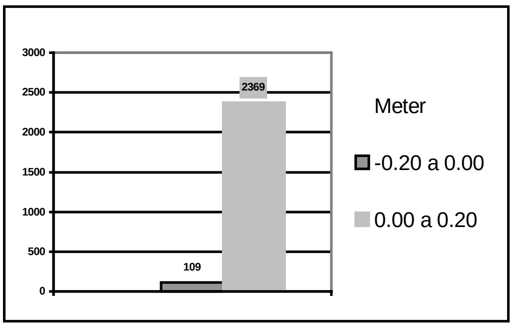

Fig. 1 Differences distribution between FFT-VK and FFT-FV

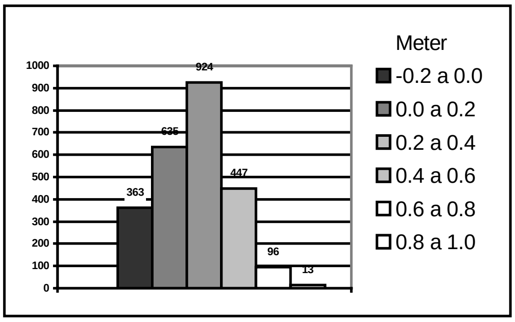

Fig. 2 Differences distribution between FFT-WG and FFT-VK

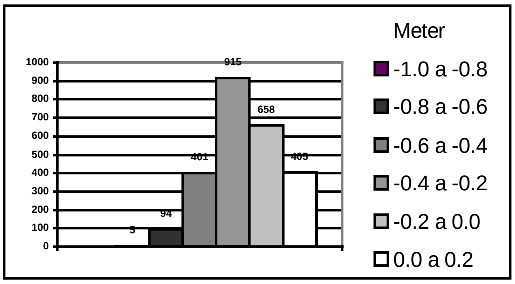

Fig. 3 Differences distribution between FFT-FV and FFT-WG

Table 2. Differences distribution between DB-VK and FFT

Experiment	RMS diff. (m)	Mean of the diff. (m)	Maximum value (m)	Minimum value (m)
DB-VK and FFT-VK	0.54	-0.01	1.88	-1.52
DB-VK and FFT-FEO	0.54	-0.01	1.88	-1.52
DB-VK and FFT-WG	0.63	0.23	1.69	-1.93

Table 2 shows that the experiment DB-VK gives the same RMS difference with FFT-VK and FFT-FEO which is a confirmation of the fact that the last two modification techniques are similar. The mean difference (third column of the Table 2) also shows that virtually no systematic difference exists between them. Again, the comparison related to FFT-WG present a RMS difference slightly greater with some systematic effect. The fact that the numerical integration (DB-VK) is quite different from FFT is not explained at the moment. Further investigation must be carried out on the issue.

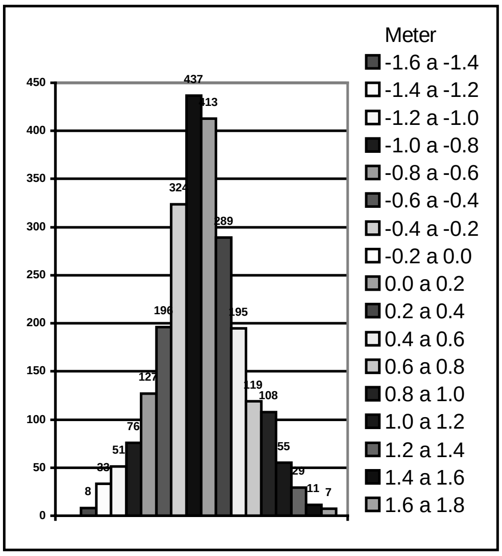

Fig. 4 Differences distribution between DB-VK and FFT-FEO

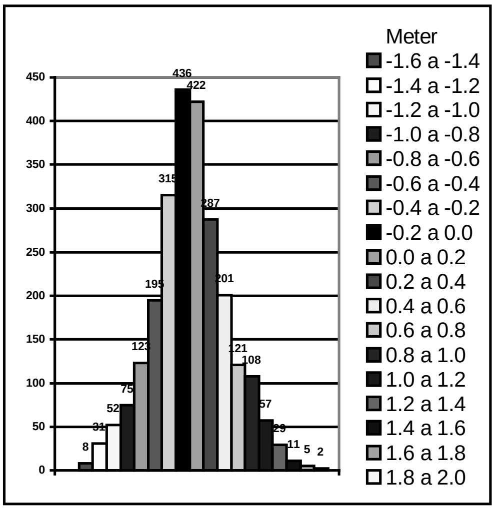

Fig. 5 Differences distribution between DB-VK e FFT-VK

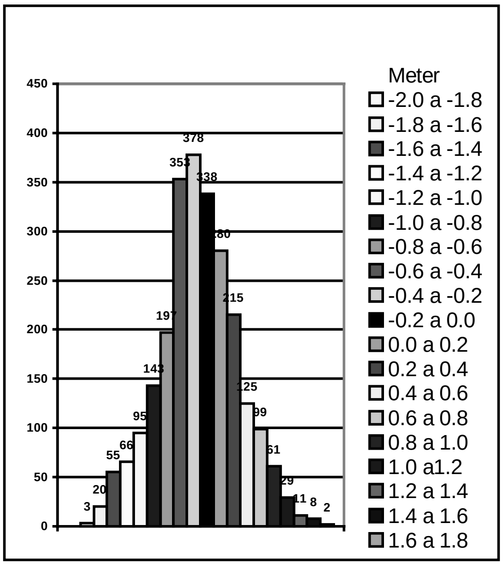

Fig. 6 Differences distribution between DB-VK e FFT-WG

Another alternative used for the evaluation was a set of 89 GPS leveled points. The results are presented in the Table 3, and Figures 7 to 10 show the distribution of the differences.

Table 3. Summary of the comparisons carried out with GPS points.

Experiment	RMS (m)	Mean (m)	Maximum (m)	Minimum (m)
DB-VK	0.70	-0.49	1.33	-1.65
FFT-VK	0.78	-0.74	-0.04	-1.57
FFT-FEO	0.77	-0.74	-0.04	-1.56
FFT-WG	0.62	-0.46	0.29	-1.47

Looking at Table 3, it can be seen that the experiment FFT-WG resulted in the smaller RMS difference (0.62 m) and also in the smaller systematic effect with respect to the others. The histograms also show a rear-random distribution.

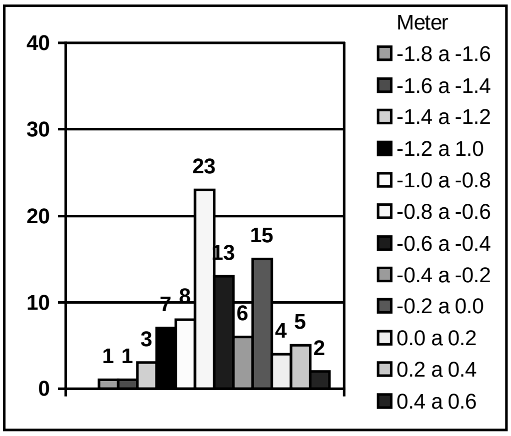

Fig. 7 Differences distribution between DB-VK and GPS points**.**

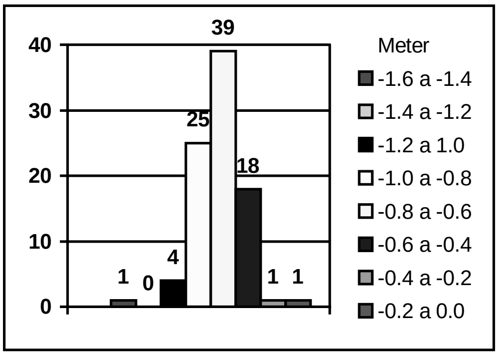

Fig. 8 Differences distribution between FFT-VK and GPS points.

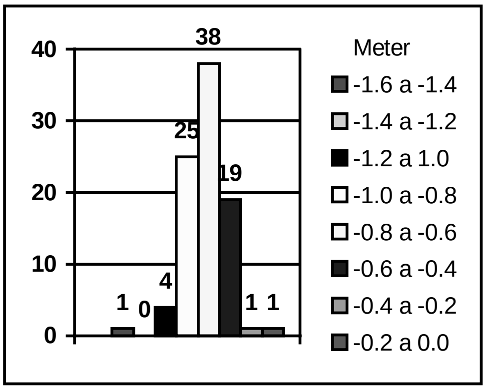

Fig. 9 Differences distribution between FFT-FEO and GPS points.

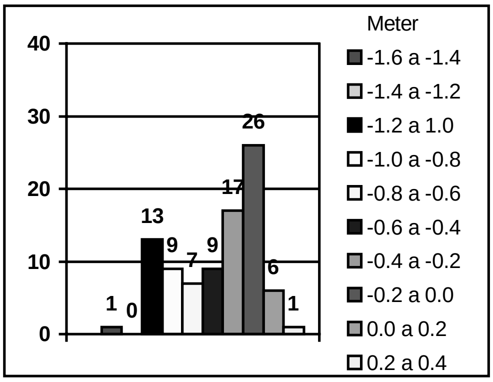

Fig. 10 Differences distribution between FFT-WG and GPS points**.**

5 Conclusions

Based on the results obtained and the analysis carried out, the following conclusions can be reached:

a) The results derived from the modifications Vaníček-Kleusberg (FFT-VK) and Featherstone et al (1998) (FFT-FEO) demonstrated to be very similar. This was expected because the modifications follow a similar principle.
b) The experiments FFT-VK and DB-VK would have to produce similar results because they are based on the same technique of modification. The only difference are the integration algorithm. Nevertheless, the results are very different and there are no explanations for that at the moment. The problem must be investigated more carefully.
c) The fact that the Wong-Gore modification showed a better agreement with the GPS points is somewhat surprising. According to Featherstone et al. (1998), the modification FFT-FEO is the one that should produce the smaller truncation error. In this case, it is the one that had to have a better agreement with the GPS points. It is well known that the GPS observations are very accurate. The leveling network in Brazil is very extensive with the origin in a tide gauge very far from the computation area. At the moment it is not easy to predict the error in the height. On the other hand, errors in the gravity anomalies or a non-homogeneous distribution can also contribute to the differences. So, the GPS points have to be used just to show a tendency, not as an absolute comparison.
d) Due to theoretical reasons a model of the geoid in the studied area has been derived from FFT-FV. The area is limited by latitudes of 19^0 and 26^0 S and by longitudes of 54^0 and 44^0 W. The model has an RMS difference of 0.78 m when compared with the GPS and a relative accuracy of 0.4 cm/km or 4 PPM, estimated from 15 pairs of points.

6 Acknowledgments

We acknowledge Will Featherstone and Michael Sideris for providing a copy of the 1D-FFT software with modification of Stokes's kernel implemented.

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